

# ON COMPLEX SUPERMANIFOLDS WITH TRIVIAL CANONICAL BUNDLE

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## Abstract

We give an algebraic characterisation for the triviality of the canonical bundle of a complex supermanifold in terms of a certain Batalin-Vilkovisky superalgebra structure. As an application, we study the Calabi-Yau case, in which an explicit formula in terms of the Levi-Civita connection is achieved. Our methods include the use of complex integral forms and the recently developed theory of superholonomy.

## 1 Introduction

The structure of a Batalin-Vilkovisky (BV) algebra was first found in the perturbative solutions of the quantum master equation, which is demanded by BRS invariance in the quantisation of gauge theories [BV81]. Through a formal analogy with the Maurer-Cartan equation of complex deformations on a complex manifold  $M$ , a certain BV structure also arises as a necessary condition for the existence of infinitesimal solutions [BK98], and is tightly related to the canonical bundle of  $M$  being trivial.

In this article, we are interested in the case of a complex supermanifold  $M$  of dimension  $\dim M = (\dim M)_{\overline{0}} | (\dim M)_{\overline{1}} = n | m$ .  $M$  is assumed connected and, as usual, we let  $M_{\overline{0}}$  denote the underlying complex manifold, and  $\mathcal{O}_M$  the sheaf of holomorphic superfunctions. By a slight abuse of notation, the superalgebra of global sections shall be denoted by the same symbol. We define the *canonical bundle* of  $M$  to be the Berezinian of the complex cotangent sheaf  $\mathrm{Ber} M := \mathrm{Ber}(\mathcal{T}^{1,0} M)^*$ , which transforms through the superdeterminant  $\mathrm{sdet} d\varphi$  under a holomorphic change of coordinates  $\varphi$ , and, in the case of an ordinary complex manifold, reduces to the classical canonical bundle of top degree holomorphic forms. Moreover, we consider the following sheaf of  $(0, q)$ -forms with values in holomorphic multivector fields.

$$(1) \quad \Omega^{0,*} \left( \bigwedge^* \mathcal{T}^{1,0} M \right) := \bigoplus_{q=0}^{\infty} \bigoplus_{p=0}^{\infty} \Omega^{0,q} \left( \bigwedge^p \mathcal{T}^{1,0} M \right)$$

Our first main theorem, to be stated next, gives an algebraic characterisation for the triviality of the canonical bundle, that is  $\mathrm{Ber} M \cong \mathcal{O}_M^{1|0}$  for  $m$  even and  $\mathrm{Ber} M \cong \mathcal{O}_M^{0|1}$  for  $m$  odd, respectively.

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**Theorem A.** *Let  $M$  be a simply connected complex supermanifold. Then the superalgebra  $\Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M)$  carries the structure of a differential Gerstenhaber-Batalin-Vilkovisky (dGBV) superalgebra strongly compatible with the Schouten-Nijenhuis bracket if and only if  $\text{Ber}M$  is trivial.*

*In this case, there is a 1:1-correspondence between trivialising homogeneous global sections  $\omega \in \text{Ber}M$  (up to a complex constant) and such dGBV-structures.*

For the case  $m = (\dim M)_{\overline{1}} = 0$  of a complex manifold, this theorem is a classical result [Sch98]. The superisation in the present form is the major achievement of this article. It only became available thanks to the recently developed superholonomy theory ([Gal09, Gro14b, Gro16]). We remark that Thm. A justifies, a posteriori, our definition of the canonical bundle of a complex supermanifold.

We comment on notions and structure of the proof, and of the article. To begin with, we state the general definition of a dGBV superalgebra in the first part of Sec. 2. In particular, we carefully explain our conventions used. In the second part of that section, we treat the case of dGBV structures on the superalgebra (1). In particular, we analyse compatibility conditions with the Schouten-Nijenhuis bracket, which is a natural extension of the vector field bracket.

One direction of the proof of Thm. A is established as follows. Starting with a trivialisation of the canonical bundle, the dGBV superalgebra structure is obtained through a super extension of an identity stated in [BK98] which, in turn, generalises a lemma proved independently by both Tian [Tia87] and Todorov [Tod89]. This is the subject matter of Sec. 3 with the main result contained in Prp. 3.4. For this implication, it is not necessary to assume that  $M$  be simply connected.

Conversely, we construct and study a connection associated with a given dGBV structure in Sec. 4, which turns out to be flat. The remaining direction of Thm. A is then obtained by means of the aforementioned superholonomy theory. More precisely, the Holonomy Principle applied to the present situation provides us with a global section  $\omega \in \text{Ber}M$  as advertised, parallel with respect to the aforementioned connection (Prp. 4.3).

In Sec. 5, we study semi-Riemannian supermanifolds with holonomy contained in some special unitary supergroup. This condition will be referred to as 'Calabi-Yau', even though the Calabi-Yau theorem does not generalise to supergeometry (we refer to the remarks in that section for details). The Levi-Civita connection on a Calabi-Yau supermanifold induces a connection on  $\text{Ber}M$  with trivial holonomy and, therefore, a parallel trivialising section. In this case, all constructions are natural, as established in our second main theorem as follows. A more precise version is provided as Thm. 5.7 below.

**Theorem B.** *The flat connection on  $\text{Ber}M$  constructed in Sec. 4 with respect to the dGBV structure of Sec. 3 on a Calabi-Yau supermanifold coincides with the one induced by the Levi-Civita connection. In particular, the dGBV structure can be explicitly expressed in terms of the Levi-Civita connection.*

Throughout this article, we have tried not to disrupt the main flow of argument longer than necessary. Mainly, but not only, for this reason we have compiled an independent appendix, App. A, on selected elements of complex supergeometry, with definitions, conventions and results contained for easy reference in the main text. Topics include a synopsis on supermanifolds and vector bundles, definition and properties

of the Schouten-Nijenhuis bracket, parallel transport and superholonomy and, finally, an exposition on the canonical bundle and integral forms.

## 2 Batalin-Vilkovisky Superstructures

This section introduces the general notions of BV and dGBV superalgebras first, followed by a brief account on algebraic properties in the cases based on our main example (1) of a superalgebra.

The structures presented are meant to be superisations of their classical counterparts. We remark that several variants of the latter exist in the literature, of which we mainly follow [Man99], [Huy05]. If not stated otherwise, a superalgebra will be either real or complex, and the notion of linearity refers to the choice of scalars.

On the superalgebra (1) which is supposed to carry a BV structure, one observes that there are two sorts of  $\mathbb{Z}_2$ -gradings involved: The cohomological degree  $q + p$  as well as the parity ('super degree') of objects induced by the parities of vectors and covectors. In general, such a situation can be modelled by a *graded superalgebra*, which is a vector space with a bigrading and a corresponding multiplication. Another possibility is to combine the two degrees to a single one and consider ordinary super (i.e.  $\mathbb{Z}_2$ -graded) algebras. It has been argued in [DM99] that both approaches are equivalent, although leading to different signs. They are referred to as "point of view I" and "point of view II" in that reference.

Throughout this article, we shall consistently adopt the first point of view. This affects, in particular, the rest of this section but also parts of the appendix, App. A. We use the following convention: The 'super degree' of an object  $X$  is denoted  $|X| \in \mathbb{Z}_2$ , while its cohomological degree is referred to as  $\deg X \in \mathbb{Z}_2$ . To give an example, we state the supercommutativity rule for a graded superalgebra with this notation.

$$a \cdot b = (-1)^{\deg(a)\deg(b)+|a||b|} b \cdot a$$

**Definition 2.1.** *We call a linear map  $f$  between graded superalgebras deg-odd, if it is odd (i.e. parity-reversing) with respect to the cohomological degree and even (i.e. parity-preserving) with respect to the super degree.*

We proceed with our main definitions.

**Definition 2.2.** *A Batalin-Vilkovisky (BV) superalgebra is a pair  $(A, \Delta)$ , where  $A$  is a supercommutative graded complex superalgebra, and  $\Delta : A \rightarrow A$  is a deg-odd complex linear map such that, for every  $\alpha \in A$ , the map*

$$\delta_\alpha : A \rightarrow A, \quad \beta \mapsto (-1)^{\deg(\alpha)} \Delta(\alpha \cdot \beta) - (-1)^{\deg(\alpha)} \Delta(\alpha) \cdot \beta - \alpha \cdot \Delta(\beta)$$

*is a derivation of degree  $\deg(\alpha) + 1$  and super degree the same as  $\alpha$ . In other words, it satisfies*

$$\delta_\alpha(\beta \cdot \gamma) = \delta_\alpha(\beta) \cdot \gamma + (-1)^{(\deg(\alpha)+1)\deg(\beta)+|\alpha||\beta|} \beta \cdot \delta_\alpha(\gamma)$$

**Definition 2.3.** *We say that a BV superalgebra  $(A, \Delta)$  is compatible with a bracket, i.e. with a complex bilinear map  $[\cdot, \cdot] : A \times A \rightarrow A$  if  $[\alpha, \beta] = -\delta_\alpha(\beta)$  for all  $\alpha, \beta \in A$ .*

We remark that the sign in the previous definition may be replaced by any constant  $c \in \mathbb{C}^*$  upon redefining  $\Delta$  accordingly. The present choice is consistent with our conventions for the operators occurring in the generalised Tian-Todorov lemma, Lem. 3.3, below.

**Definition 2.4.** A BV superalgebra  $(A, \Delta)$  is called Gerstenhaber-Batalin-Vilkovisky (GBV) if  $\Delta \circ \Delta = 0$ .

The following statement is a straightforward consequence of the axioms.

**Lemma 2.5.** On a GBV superalgebra  $(A, \Delta)$  which is compatible with a bracket  $[\cdot, \cdot]$ , the following equation is satisfied.

$$-\Delta([\alpha, \beta]) = -[\Delta(\alpha), \beta] + (-1)^{\deg(\alpha)} [\alpha, \Delta(\beta)]$$

*Proof.* Using  $\Delta^2 = 0$ , the compatibility equation  $[\alpha, \beta] = -\delta_\alpha(\beta)$  with  $\alpha$  replaced by  $\Delta(\alpha)$ , and with  $\beta$  replaced by  $\Delta(\beta)$ , respectively, reads as follows.

$$\begin{aligned} -(-1)^{\deg(\alpha)} \Delta(\Delta(\alpha) \cdot \beta) &= -[\Delta(\alpha), \beta] + \Delta(\alpha) \cdot \Delta(\beta) \\ -\Delta(\alpha \cdot \Delta(\beta)) &= (-1)^{\deg(\alpha)} [\alpha, \Delta(\beta)] - \Delta(\alpha) \cdot \Delta(\beta) \end{aligned}$$

Similarly, applying  $\Delta$  on both sides of the compatibility equation, we obtain

$$-\Delta([\alpha, \beta]) = -(-1)^{\deg(\alpha)} \Delta(\Delta(\alpha) \cdot \beta) - \Delta(\alpha \cdot \Delta(\beta))$$

With the previous formulas, the statement follows.  $\square$

**Definition 2.6.** A differential Gerstenhaber-Batalin-Vilkovisky (dGBV) superalgebra is a triple  $(A, \Delta, d)$  such that  $(A, \Delta)$  is a GBV algebra and  $d$  is a deg-odd complex linear derivation of degree 1, i.e. such that

$$d(\alpha \cdot \beta) = d(\alpha) \cdot \beta + (-1)^{\deg(\alpha)} \alpha \cdot d(\beta)$$

and, in addition, such that  $d \circ d = 0$  and  $d \circ \Delta + \Delta \circ d = 0$ .

## 2.1 The Schouten-Nijenhuis Case

Having introduced the general definitions, we now consider dGBV superalgebras of the form  $(\Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M), \Delta, \overline{\partial})$  which are compatible with the Schouten-Nijenhuis bracket as defined in Def. A.2. The latter property we shall simply refer to as *compatible* in the following. The only datum to be specified in the case at hand is an appropriate operator  $\Delta : \Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M) \rightarrow \Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M)$ . By the following simple observation, the definition of such an operator on the entire superalgebra is highly redundant.

**Lemma 2.7.** Let  $(\Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M), \Delta, \overline{\partial})$  be a dGBV superalgebra compatible with the Schouten-Nijenhuis bracket. Then  $\Delta$  is uniquely determined by its values on elements of the following types, for every open subset  $U \subseteq M_{\overline{0}}$ : Functions  $f \in \mathcal{O}_M(U)$ , tangent vectors  $X \in \mathcal{T}^{1,0}M(U)$ , and forms  $\lambda \in \Omega^{0,1}M(U)$ .

*Proof.* Locally, that is upon restriction to a sufficiently small open subset  $U \subseteq M_{\bar{0}}$ , every element of  $\Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M)$  can be written as a polynomial in functions, tangent vectors and  $(0,1)$ -forms. By linearity, it suffices to consider monomials. Having specified  $\Delta$  restricted to the according subspaces, the original operator is recursively determined by the Schouten-Nijenhuis bracket compatibility condition in the following form.

$$\Delta(\alpha \cdot \beta) = -(-1)^{\deg(\alpha)} [\alpha, \beta] + \Delta(\alpha) \cdot \beta + (-1)^{\deg(\alpha)} \alpha \cdot \Delta(\beta)$$

□

The following condition is the one arising in Thm. A.

**Definition 2.8.** A dGBV superalgebra  $(\Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M), \Delta, \bar{\partial})$  is called strongly compatible (with the Schouten-Nijenhuis bracket) if it is compatible and  $\Delta$  is a direct sum of operators

$$\Delta^{p,q} : \Omega^{0,q} \left( \bigwedge^p \mathcal{T}^{1,0}M \right) \rightarrow \Omega^{0,q} \left( \bigwedge^{p-1} \mathcal{T}^{1,0}M \right)$$

In other words, we demand  $\Delta$  to treat the  $\mathbb{Z}$ -degree (that we otherwise do not consider) analogous to the Schouten-Nijenhuis bracket. The characterisation in terms of Lem. 2.7 reads as follows.

**Corollary 2.9.** A compatible dGBV superalgebra is strongly compatible if and only if  $\Delta$  has the following properties.  $\Delta(f) = 0$  for  $f \in \mathcal{O}_M(U)$  and  $\Delta(X) \in \mathcal{O}_M(U)$  for  $X \in \mathcal{T}^{1,0}M(U)$  and  $\Delta(\lambda) = 0$  for  $\lambda \in \Omega^{0,1}M(U)$ , for every open subset  $U \subseteq M_{\bar{0}}$ .

As to be elaborated now, the operator  $\Delta$  of a compatible dGBV superalgebra can be projected such as to give rise to a strongly compatible dGBV superalgebra. Let  $\Delta : \Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M) \rightarrow \Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M)$  be a complex linear operator. We let  $\tilde{\Delta} : \Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M) \rightarrow \Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M)$  denote the direct sum of projections

$$\tilde{\Delta}^{p,q} := \Pi_{p-1,q} \circ \Delta|_{\Omega^{0,q}(\bigwedge^p \mathcal{T}^{1,0}M)}, \quad \Pi_{p-1,q} := \Pi_{\Omega^{0,q}(\bigwedge^{p-1} \mathcal{T}^{1,0}M)}$$

corresponding to the decomposition (1).

**Lemma 2.10.** Let  $(\Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M), \Delta, \bar{\partial})$  be a compatible dGBV superalgebra. Then  $(\Omega^{0,*}(\bigwedge^* \mathcal{T}^{1,0}M), \tilde{\Delta}, \bar{\partial})$  is a strongly compatible dGBV superalgebra.

*Proof.* The statement will be clear from the following three properties of the operator  $\tilde{\Delta}$  that we shall establish subsequently: It is compatible with the Schouten-Nijenhuis bracket, anticommutes with  $\bar{\partial}$  and squares to zero.

Let  $\alpha \in \Omega^{0,r}(\bigwedge^s \mathcal{T}^{1,0}M)$  and  $\beta \in \Omega^{0,q}(\bigwedge^p \mathcal{T}^{1,0}M)$ . Applying  $\Pi_{s+p-1,r+q}$  to both sides of the equation of Schouten-Nijenhuis compatibility with respect to  $\Delta$ , we obtain

$$[\alpha, \beta] = (-1)^{\deg(\alpha)} \tilde{\Delta}(\alpha \cdot \beta) - (-1)^{\deg(\alpha)} \Pi_{s+p-1,r+q}(\Delta(\alpha) \cdot \beta) - \Pi_{s+p-1,r+q}(\alpha \cdot \Delta(\beta))$$

from which compatibility of  $\tilde{\Delta}$  is clear.

The equation  $\bar{\partial} \circ \tilde{\Delta} + \tilde{\Delta} \circ \bar{\partial} = 0$  follows from the corresponding property of  $\Delta$  by an analogous calculation.

It remains to show  $\tilde{\Delta}^2 = 0$ . Obviously, this is true upon application to  $\alpha \in \Omega^{0,r}M = \Omega^{0,r}(\bigwedge^0 \mathcal{T}^{1,0}M)$  since, for such  $\alpha$ , already  $\tilde{\Delta}(\alpha) = 0$  holds by definition of  $\tilde{\Delta}$ . For

$\alpha \in \Omega^{0,r}(\bigwedge^1 \mathcal{T}^{1,0}M)$ , we have that  $\tilde{\Delta}(\alpha) \in \Omega^{0,r}M$  and, therefore, also  $\tilde{\Delta}^2(\alpha) = 0$ . By induction on  $s$ , we now assume that  $\tilde{\Delta}^2(\alpha) = 0$  for  $\alpha \in \Omega^{0,r}(\bigwedge^{s-1} \mathcal{T}^{1,0}M)$  with  $r \in \mathbb{N}$  arbitrary. Let  $\alpha \in \Omega^{0,r}(\bigwedge^s \mathcal{T}^{1,0}M)$ . Locally we may write  $\alpha = \beta \cdot \gamma$  with  $\beta \in \Omega^{0,r}(\bigwedge^1 \mathcal{T}^{1,0}M)$  and  $\gamma \in \Omega^{0,0}(\bigwedge^{s-1} \mathcal{T}^{1,0}M)$ . Using Schouten-Nijenhuis compatibility, we calculate

$$\begin{aligned} \tilde{\Delta}^2(\alpha) &= \tilde{\Delta} \left( (-1)^{\deg(\beta)} [\beta, \gamma] + \tilde{\Delta}(\beta) \cdot \gamma + (-1)^{\deg(\beta)} \beta \cdot \tilde{\Delta}(\gamma) \right) \\ &= (-1)^{\deg(\beta)} \tilde{\Delta} [\beta, \gamma] + (-1)^{\deg(\beta)-1} \left[ \tilde{\Delta}(\beta), \gamma \right] + (-1)^{\deg(\beta)-1} \tilde{\Delta}(\beta) \cdot \tilde{\Delta}(\gamma) \\ &\quad + [\beta, \tilde{\Delta}(\gamma)] + (-1)^{\deg(\beta)} \tilde{\Delta}(\beta) \cdot \tilde{\Delta}(\gamma) \\ &= (-1)^{\deg(\beta)} \tilde{\Delta} [\beta, \gamma] + (-1)^{\deg(\beta)-1} \left[ \tilde{\Delta}(\beta), \gamma \right] + [\beta, \tilde{\Delta}(\gamma)] \\ &= \Pi_{s-1,r} \left( (-1)^{\deg(\beta)} \Delta [\beta, \gamma] + (-1)^{\deg(\beta)-1} [\Delta(\beta), \gamma] + [\beta, \Delta(\gamma)] \right) \end{aligned}$$

Now, performing the first two steps in this calculation in reverse order, with  $\tilde{\Delta}$  replaced by  $\Delta$ , we find

$$\tilde{\Delta}^2(\alpha) = \Pi_{s-1,r} \circ \Delta^2(\alpha) = 0$$

which was to be shown.  $\square$

### 3 A Generalised Tian-Todorov Formula

This section establishes one direction of Thm. A by means of a generalised Tian-Todorov lemma. We shall use a suitable generalisation of the classical operator  $\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}$  (with  $q = 0$ ) on a complex manifold. This operator acts on integral forms  $\partial : I^{n-p} \rightarrow I^{n-p+1}$  on  $M$  rather than on differential forms and, modulo some identifications, was already studied in [Man88]. We refer to the appendix, Sec. A.4, for details, and be very brief at this point. A local formula, in terms of coordinates  $(\xi^k)$ , reads as follows.

$$\begin{aligned} \partial \left( f \cdot \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \otimes [d\xi] \right) \\ := \sum_{i=1}^{n+m} (-1)^{M_i} \frac{\partial f}{\partial \xi^i} \cdot \left( \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^{i-1}} \wedge \widehat{\frac{\partial}{\partial \xi^i}} \wedge \frac{\partial}{\partial \xi^{i+1}} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \right) \otimes [d\xi] \end{aligned}$$

where  $M_i$  is the sign arising from moving  $\frac{\partial}{\partial \xi^i}$  to the front. We shall denote the extension (20) to an operator

$$\partial : \Omega^{0,q}(I^{n-p}) = \Omega^{0,q}M \otimes I^{n-p} \longrightarrow \Omega^{0,q}M \otimes I^{n-p+1} = \Omega^{0,q}(I^{n-p+1})$$

by the same symbol.

For the rest of this section, we assume that  $\text{Ber}M$  is trivial, and we fix a global trivialising section  $\omega \in \text{Ber}M$ . Every  $\alpha \in \text{Ber}M$  is then uniquely determined by a unique superfunction  $f \in \mathcal{O}_M$  such that  $\alpha = f \cdot \omega$ . Without loss of generality,  $\omega$  may (and will) be chosen homogeneous, that is of parity  $|\omega| = (\dim M)_{\overline{1}}$ . Locally, with respect to coordinates  $(\xi^k)$ , we can write  $\omega = h \cdot [d\xi]$  with  $h$  a local, even, invertible

and holomorphic superfunction. Moreover, a choice of  $\omega$  defines an isomorphism (even or odd)

$$\eta : \bigwedge^p \mathcal{T}^{1,0} M \rightarrow I^{n-p}, \quad \eta(v_1 \wedge \dots \wedge v_p) := (v_1 \wedge \dots \wedge v_p) \otimes \omega$$

The map  $\eta$  induces canonical isomorphisms, denoted by the same symbol:

$$\eta : \Omega^{0,q} \left( \bigwedge^p \mathcal{T}^{1,0} M \right) \longrightarrow \Omega^{0,q} (I^{n-p})$$

On both sides, there is the  $\bar{\partial}$ -operator (14). By construction, it acts only on the form part, whence we obtain

$$(2) \quad \eta \circ \bar{\partial} = \bar{\partial} \circ \eta$$

**Definition 3.1.** We define the operator  $\Delta^\omega$  as follows.

$$\Delta^\omega : \Omega^{0,q} \left( \bigwedge^p \mathcal{T}^{1,0} M \right) \rightarrow \Omega^{0,q} \left( \bigwedge^{p-1} \mathcal{T}^{1,0} M \right), \quad \Delta^\omega := \eta^{-1} \circ \partial \circ \eta$$

**Lemma 3.2.** The operator  $\Delta^\omega$  satisfies the following properties, for every open subset  $U \subseteq M_{\bar{0}}$ . It vanishes on functions  $f \in \mathcal{O}_M(U)$  and forms  $\lambda \in \Omega^{0,1} M(U)$ . Applied to vectors  $X \in \mathcal{T}^{1,0} M(U)$ , it takes values in functions  $\Delta^\omega(X) \in \mathcal{O}_M(U)$ . The local formula reads  $\Delta^\omega(\partial_{\xi^k}) = \frac{\partial h}{\partial \xi^k} h^{-1}$ , where  $\omega = h \cdot [d\xi]$ .

*Proof.* This statement is clear by the definition of  $\partial$  (and  $\eta$ ), with the calculation for coordinate vector fields as follows.

$$\Delta^\omega(\partial_{\xi^k}) = \eta^{-1} \partial (h \cdot \partial_{\xi^k} \otimes [d\xi]) = \eta^{-1} \left( \frac{\partial h}{\partial \xi^k} \cdot [d\xi] \right) = \frac{\partial h}{\partial \xi^k} h^{-1}$$

□

The identity of the following result was stated in [BK98] for the case of a classical complex manifold with trivial canonical bundle. This, in turn, generalises a lemma proved independently by both Tian [Tia87] and Todorov [Tod89]. We will, therefore, refer to the following supergeometric generalisation also as "generalised Tian-Todorov lemma".

**Lemma 3.3** (Tian-Todorov). Let  $\alpha \in \Omega^{0,p}(\bigwedge^s \mathcal{T}^{1,0} M)$  and  $\beta \in \Omega^{0,q}(\bigwedge^r \mathcal{T}^{1,0} M)$ . Then the operator  $\Delta^\omega$  just defined is compatible with the Schouten-Nijenhuis bracket  $[\cdot, \cdot]$  in the following sense.

$$-[\alpha, \beta] = (-1)^{\deg(\alpha)} \Delta^\omega(\alpha \wedge \beta) - (-1)^{\deg(\alpha)} \Delta^\omega(\alpha) \wedge \beta - \alpha \wedge \Delta^\omega(\beta)$$

The Schouten-Nijenhuis bracket in the present context is defined in Def. A.2 below, while  $\deg(\alpha) = p + s$  denotes the cohomological degree of  $\alpha$  as in Sec. 2.

*Proof.* The local formula of Lem. 3.2 for  $\Delta$  applied to tangent vectors generalises as follows.

$$\begin{aligned} & \Delta(f \cdot \partial_{\xi^1} \wedge \dots \wedge \partial_{\xi^r}) \\ &= \sum_i (-1)^{(i-1) + |\xi^i|(|f| + \sum_{l=1}^{i-1} |\xi^l|)} \frac{\partial(f \cdot h)}{\partial \xi^i} h^{-1} (\partial_{\xi^1} \wedge \dots \wedge \widehat{\partial_{\xi^i}} \wedge \dots \wedge \partial_{\xi^r}) \end{aligned}$$



As usual, the hat symbol means omission. The Tian-Todorov formula, for the case  $p = q = 0$ , is established from this formula and the definition of the Schouten-Nijenhuis bracket, Def. A.2, in a lengthy but straightforward calculation.

In the second step, we observe that the general case can be deduced from the case  $p = q = 0$  already established. Again, this is a direct calculation in local coordinates, unwinding the definitions.  $\square$

**Proposition 3.4.** *The triple  $(\Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M), \Delta^\omega, \bar{\partial})$  is a dGBV superalgebra strongly compatible with the Schouten-Nijenhuis bracket.*

We remark that, thanks to this result together with Lem. 2.7, the operator  $\Delta^\omega$  is completely determined by its values stated in Lem. 3.2.

*Proof.* By construction,  $\Delta^\omega$  is a deg-odd linear map.  $(A, \Delta^\omega)$  is a BV superalgebra by Lem. A.5, together with the Tian-Todorov lemma, Lem. 3.3. Compatibility holds true by the same formula. It remains to show that, together with  $\bar{\partial}$ , it is in fact dGBV. Strong compatibility is then clear by construction of  $\Delta^\omega$ . In fact, by (21), we see that  $\Delta^2 = \eta^{-1} \circ \partial^2 \circ \eta = 0$ . Moreover, the formula  $\bar{\partial} \circ \Delta^\omega + \Delta^\omega \circ \bar{\partial} = 0$  holds true since  $\bar{\partial}$  commutes with  $\eta$  and anticommutes with  $\partial$ , by (2) and Lem. A.13, respectively.  $\square$

## 4 Flat Connections on the Canonical Bundle

Having successfully translated triviality of the canonical bundle into existence of a dGBV structure, we now turn to the converse, thus establishing the remaining direction of Thm. A. In the following, we assume that  $M$  is a complex supermanifold equipped with an operator  $\Delta : \Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M) \rightarrow \Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M)$ , such that the triple  $(\Omega^{0,*}(\wedge^* \mathcal{T}^{1,0}M), \Delta, \bar{\partial})$  is a dGBV superalgebra strongly compatible with the Schouten-Nijenhuis bracket. We shall construct a flat connection on  $\text{Ber}M$  and establish triviality by means of a parallel section obtained by superholonomy theory, which is sketched in the appendix. To be precise, we shall need the Holonomy Principle, Thm. A.8, together with the Ambrose-Singer Theorem, cf. (19). At this point, the assumption that  $M$  be simply-connected becomes important.

**Lemma 4.1.** *Let  $(\xi^k)$  denote complex coordinates of  $M$  with induced real coordinates (11). Then the prescription*

$$(\nabla^\xi)_{\partial_{\xi_R^k}}[d\xi] := -\Delta(\partial_{\xi^k}) \cdot [d\xi], \quad (\nabla^\xi)_{\partial_{\xi_I^k}}[d\xi] := -i \cdot \Delta(\partial_{\xi^k}) \cdot [d\xi]$$

*together with the Leibniz rule defines a connection on  $\text{Ber}M$ .*

We remark that, as usual, this definition is meant to be done with respect to any coordinate system, and the resulting expressions are claimed to transform such as to constitute a global object. By construction, this is then a connection. Moreover, we remind the reader of our convention that a connection is always real, with complex linearity if present considered as an extra structure. In the case at hand, we find

$$(\nabla^\xi)_{JX}[d\xi] = i \cdot (\nabla^\xi)_X[d\xi]$$

which, upon complex linear extension, implies

$$(\nabla^\xi)_{\partial_{\xi^k}}[d\xi] = (\nabla^\xi)_{\partial_{\xi_R^k}}[d\xi] = -\Delta(\partial_{\xi^k}) \cdot [d\xi]$$



*Proof.* By the preceding remark, we may work completely in the complex picture. We need to show the following analogon of (17) under a coordinate transformation  $\varphi : \zeta \rightarrow \xi$ .

$$\varphi^*((\nabla^\xi)_{\partial_{\xi^k}}[d\xi]) = (\nabla^\zeta)_{\varphi^*\partial_{\xi^k}}\varphi^*[d\xi]$$

We calculate the right hand side, using (10) and Leibniz' rule.

$$\begin{aligned} (\nabla^\zeta)_{\varphi^*\partial_{\xi^k}}\varphi^*[d\xi] &= (-1)^{(m+k)m}(d\varphi^{-1})^m_k \partial_{\zeta^m}(\text{sdet}d\varphi) \cdot [d\zeta] \\ &\quad - (-1)^{(m+k)m}(d\varphi^{-1})^m_k \text{sdet}d\varphi \cdot \Delta(\partial_{\zeta^m}) \cdot [d\zeta] \\ &= \partial_{\zeta^m}((d\varphi^{-1})^m_k \text{sdet}d\varphi) [d\zeta] - \partial_{\zeta^m}(d\varphi^{-1})^m_k \text{sdet}d\varphi \cdot [d\zeta] \\ &\quad - (-1)^{(m+k)m}(d\varphi^{-1})^m_k \text{sdet}d\varphi \cdot \Delta(\partial_{\zeta^m}) \cdot [d\zeta] \end{aligned}$$

By Lem. A.1, the first term vanishes. Further utilising the compatibility equation with the Schouten-Nijenhuis bracket, we further calculate

$$\begin{aligned} (\nabla^\zeta)_{\varphi^*\partial_{\xi^k}}\varphi^*[d\xi] &= -(\partial_{\zeta^m}(d\varphi^{-1})^m_k + \Delta(\partial_{\zeta^m}) \cdot (d\varphi^{-1})^m_k) \text{sdet}d\varphi \cdot [d\zeta] \\ &= -\Delta(\partial_{\zeta^m} \cdot (d\varphi^{-1})^m_k) \text{sdet}d\varphi \cdot [d\zeta] \\ &= -\Delta(\varphi^\sharp \circ \partial_{\xi^k} \circ (\varphi^{-1})^\sharp) \varphi^*[d\xi] \\ &= -\varphi^* \Delta(\partial_{\xi^k}) \varphi^*[d\xi] \\ &= \varphi^*((\nabla^\xi)_{\partial_{\xi^k}}[d\xi]) \end{aligned}$$

thus proving well-definedness.  $\square$

**Lemma 4.2.** *The connection from Lem. 4.1 is flat (has vanishing curvature).*

*Proof.* We calculate

$$\begin{aligned} R(\partial_{\xi^l}, \partial_{\xi^m})[d\xi] &= \nabla_{\partial_{\xi^l}} \nabla_{\partial_{\xi^m}}[d\xi] - (-1)^{lm} \nabla_{\partial_{\xi^m}} \nabla_{\partial_{\xi^l}}[d\xi] \\ &= -\nabla_{\partial_{\xi^l}}(\Delta(\partial_{\xi^m}) \cdot [d\xi]) + (-1)^{lm} \nabla_{\partial_{\xi^m}}(\Delta(\partial_{\xi^l})[d\xi]) \\ &= \left( -\partial_{\xi^l}(\Delta(\partial_{\xi^m})) + (-1)^{lm} \partial_{\xi^m}(\Delta(\partial_{\xi^l})) \right) [d\xi] \\ &\quad + \left( -(-1)^{lm} \Delta(\partial_{\xi^m}) \nabla_{\partial_{\xi^l}}[d\xi] + \Delta(\partial_{\xi^l}) \nabla_{\partial_{\xi^m}}[d\xi] \right) \\ &=: (1) + (2) \end{aligned}$$

where

$$(2) = \left( (-1)^{lm} \Delta(\partial_{\xi^m}) \Delta(\partial_{\xi^l}) - \Delta(\partial_{\xi^l}) \Delta(\partial_{\xi^m}) \right) [d\xi] = 0$$

Consider the term

$$(-1)^{lm} \partial_{\xi^m}(\Delta(\partial_{\xi^l})) = (-1)^{lm} [\partial_{\xi^m}, \Delta(\partial_{\xi^l})] = -[\Delta(\partial_{\xi^l}), \partial_{\xi^m}]$$

By Lem. 2.5, it equals

$$(-1)^{lm} \partial_{\xi^m}(\Delta(\partial_{\xi^l})) = [\partial_{\xi^l}, \Delta(\partial_{\xi^m})] = \partial_{\xi^l}(\Delta(\partial_{\xi^m}))$$

whence also (1) = 0 vanishes. The statement is proved.  $\square$

**Proposition 4.3.** *Let  $M$  be simply connected. Under the above hypotheses (existence of a strongly compatible dGBV structure), there exists a global parallel nonzero homogeneous and holomorphic section  $\omega \in \text{Ber}M$ , which is unique up to multiplication by a complex number.*

*Proof.* By Lem. 4.2, the connection  $\nabla$  from Lem. 4.1 is flat. Under the hypothesis of simply connectedness, the Ambrose-Singer Theorem for the superholonomy functor (cf. (19) in the appendix) implies that  $\text{Hol}_T^\nabla = \{1\}$ . By the holonomy principle, Thm. A.8, there exists a global parallel nonzero section  $\omega \in \text{Ber}M$ , which is unique upon specifying  $\omega_x \in x^*\text{Ber}M \cong \mathbb{C}$  for some (topological) point  $x \in M_{\overline{0}}$  (in the framework of [Gro14b], we consider  $S = \mathbb{R}^{0|0}$ ). In terms of coordinates we may, locally, write  $\omega = h \cdot [d\xi]$ . By construction, the superfunction  $h$  is invertible and even.

It remains to show that  $h$  is holomorphic. As  $\omega$  is parallel, we find

$$0 = \nabla_{\partial_{\xi^k}} \omega = \nabla_{\partial_{\xi^k}} (h[d\xi]) = \partial_{\xi^k}(h)[d\xi] + h \nabla_{\partial_{\xi^k}} [d\xi] = (\partial_{\xi^k}(h) - h \Delta(\partial_{\xi^k})) [d\xi]$$

Therefore

$$(3) \quad h \Delta(\partial_{\xi^k}) = \partial_{\xi^k}(h)$$

Since  $\Delta$  is, by assumption, complex linear, it follows that  $h$  and, therefore,  $\omega$  is holomorphic.  $\square$

By means of the trivialising section  $\omega \in \text{Ber}M$  provided by Prp. 4.3, we may use Def. 3.1 to define an operator, to be denoted  $\Delta^\omega$ . From the previous section, we know that  $\Delta^\omega$  constitutes a strongly compatible dGBV structure, just like  $\Delta$ . By the following statement, both structures coincide.

**Lemma 4.4.** *Let  $\omega \in \text{Ber}M$  be induced by  $\Delta$  as in Prp. 4.3. Then  $\Delta = \Delta^\omega$ .*

*Proof.* By (3), we obtain  $\Delta(\partial_{\xi^k}) = h^{-1} \partial_{\xi^k}(h)$ . The same formula holds with  $\Delta$  replaced with  $\Delta^\omega$ , by Lem. 3.2. Since both operators are, moreover, strongly compatible, they must agree (cf. Lem. 2.7).  $\square$

*Proof of Thm. A.* Assuming that  $\text{Ber}M$  is trivial, we let  $\omega \in \text{Ber}M$  denote a homogeneous trivialising global section. The operator  $\Delta^\omega$  from Def. 3.1 then induces a strongly compatible dGBV structure (cf. Prp. 3.4). We thus obtain a map  $F : \omega \mapsto \Delta^\omega$ .

On the other hand, let  $\Delta$  constitute a strongly compatible dGBV structure. By Prp. 4.3, we obtain a homogeneous trivialising section  $\omega_\Delta \in \text{Ber}M$  (up to a constant) which is parallel with respect to the connection  $\nabla^\Delta$  of Lem. 4.1. We denote this map by  $G : \Delta \mapsto \omega_\Delta$  (defined up to a constant). By Lem. 4.4, we get  $F \circ G = \text{id}_\Delta$ .

It remains to show  $G \circ F = \text{id}_\omega$  (up to a constant). Let  $\omega \in \text{Ber}M$  be trivialising and homogeneous. With respect to coordinates  $(\xi^k)$  we write, locally,  $\omega = h \cdot [d\xi]$ . Moreover, we set  $\tilde{\omega} := G \circ F(\omega)$  and, locally,  $\tilde{\omega} = \tilde{h} \cdot [d\xi] = f \cdot h \cdot [d\xi]$ . Using Lem. 3.2 and (3), we obtain

$$h^{-1} \partial_{\xi^k}(h) = \Delta^\omega(\partial_{\xi^k}) = \tilde{h}^{-1} \partial_{\xi^k}(\tilde{h}) = h^{-1} \partial_{\xi^k}(h) + f^{-1} \partial_{\xi^k}(f)$$

It follows that  $f$  is constant, which was to be shown.  $\square$

## 5 The Calabi-Yau Case

In this section, we study a particular kind of supermanifolds with trivial canonical bundle that we refer to as Calabi-Yau. To begin with, consider the following superisation of one of the equivalent characterisations of a Kähler manifold.

**Definition 5.1.** *A Kähler supermanifold is a semi-Riemannian supermanifold  $(M, g)$  such that the holonomy group functor (associated with the Levi-Civita connection) satisfies*

$$\mathrm{Hol}_x(T) \subseteq U_{p_0, q_0 | p_1, q_1}(\mathcal{O}_T)$$

for all  $T = \mathbb{R}^{0|L'}$  at some  $x \in M_{\overline{0}}$ .

Here,  $(p_0, q_0)$  and  $(p_1, q_1)$  denote the signatures of the underlying metrics in the even/even and odd/odd directions, respectively, in the definition of the unitary supergroup. It is clear that Def. 5.1 is independent of  $x$  (we remind the reader that we consider only connected supermanifolds). Considering a different point  $y$  results in the holonomy conjugated by parallel transport from  $x$  to  $y$ . By the Twofold Theorem of [Gro16], this definition is equivalent to Galaev's given in [Gal09]. By the Holonomy Principle, Thm. A.8, a Kähler supermanifold can be characterised as a Hermitian supermanifold  $(M, g)$  with a parallel complex structure  $\nabla J = 0$ .

The Calabi-Yau case is not so straightforward. Already for classical manifolds, that term is used to denote several different concepts, which are tightly related but not equivalent. This is summarised e.g. in Sec. 6.1 of [Joy00]. The various different definitions of a Calabi-Yau manifolds can all be promoted to supergeometry, but these generalisations have even less in common, as we shall presently see. To be definite, we propose the following definition which, from the point of view of holonomy theory, is the most natural one.

**Definition 5.2.** *A Calabi-Yau supermanifold is a Kähler supermanifold such that*

$$\mathrm{Hol}_x(T) \subseteq SU_{p_0, q_0 | p_1, q_1}(\mathcal{O}_T)$$

for all  $T = \mathbb{R}^{0|L'}$  at some  $x \in M_{\overline{0}}$ .

This definition may be translated to Galaev's holonomy theory by means of the aforementioned Twofold Theorem. A Calabi-Yau supermanifold in our denomination is then equivalent to what is called "special Kähler supermanifold" in [Gal09]. We state the following characterisation in the simply-connected case, which was proved as Prp. 11.1 in that reference.

**Proposition 5.3.** *Let  $(M, g)$  be a simply-connected Kähler supermanifold. Then  $(M, g)$  is Calabi-Yau if and only if the Ricci tensor vanishes.*

Def. 5.2 is the natural generalisation of the classical definition of a Calabi-Yau manifold in terms of holonomy, and Prp. 5.3 generalises a well-known classical characterisation. However, the analogy to classical geometry is not as close for the following two reasons.

First, a Calabi-Yau supermanifold is, in particular, a Kähler supermanifold. Moreover, the smooth manifold underlying a Kähler supermanifold naturally inherits a Kähler

structure. However, the classical Kähler manifold underlying a Calabi-Yau supermanifold need not be a Calabi-Yau manifold.

Second, Calabi-Yau manifolds obtained their name from Calabi's conjecture which was later proved by Yau. The analogous theorem in supergeometry is, however, wrong. There are explicit counterexamples, see [RW05]. Nevertheless, the following generalisation of a well-known result holds true.

**Proposition 5.4.** *The Levi-Civita connection of a Calabi-Yau supermanifold induces a connection  $\nabla^{\text{Ber}}$  on  $\text{Ber}M$  with trivial holonomy. There is a global parallel nowhere vanishing homogeneous and holomorphic section  $\omega \in \text{Ber}M$ , unique up to a complex number. In particular, a Calabi-Yau supermanifold has trivial canonical bundle.*

It follows at once (by Prp. 3.4) that a Calabi-Yau supermanifold bears a strongly-compatible dGBV structure. Prp. 5.4 will be established with the help of the following two lemmas. In the first, we construct the connection  $\nabla^{\text{Ber}}$ , while the second is concerned about its parallel transports.

**Lemma 5.5.** *A connection  $\nabla$  on the tangent bundle of a complex supermanifold, which satisfies  $\nabla J = 0$ , induces a connection on  $\text{Ber}M$ , denoted  $\nabla^{\text{Ber}}$ , through the following local definition in coordinates  $(\xi^k)$ .*

$$\nabla_{\partial_{\xi^l}}^{\text{Ber}}[d\xi] := -\text{str} \left( j \circ (\nabla^\xi)_{\frac{\partial}{\partial \xi^l}} \circ j^{-1} \right) \cdot [d\xi]$$

Here, conjugation with  $j$  translates  $\nabla$  to a connection (15) on  $\mathcal{T}^{1,0}M$ , which the definition requires to be complex linear. The condition  $\nabla J = 0$  is necessary and sufficient for that property.

*Proof.* We need to show that the local definition given behaves correctly under a coordinate transformation  $\varphi : \zeta \rightarrow \xi$ . This proof largely parallels that of Lem. 4.1. Abbreviating  $\nabla := \nabla^{\text{Ber}}$ , one side of the equation to be established may be expressed as

$$(\nabla^\zeta)_{\varphi^* \partial_{\xi^k}} \varphi^*[d\xi] = - \left( \partial_{\zeta^m} (d\varphi^{-1})^m_k + \text{str} \left( j \circ (\nabla^\zeta)_{\partial_{\zeta^m}} \circ j^{-1} \right) \cdot (d\varphi^{-1})^m_k \right) \text{sdet} d\varphi \cdot [d\xi]$$

whereas the other side reads as follows.

$$\varphi^*((\nabla^\xi)_{\partial_{\xi^k}}[d\xi]) = -\text{str}(j \circ \varphi^\#(\Gamma^\xi)_{i_k} \circ j^{-1}) \cdot \varphi^*[d\xi]$$

The Christoffel symbols transform according to Lem. A.6. Further utilising Jacobi's formula (9), together with Lem. A.1, one arrives at the same expression as before, thus showing well-definedness.  $\square$

**Lemma 5.6.** *Let  $P_\gamma$  denote parallel transport with respect to the connection  $\nabla$  along some  $S$ -path  $\gamma$ . Then*

$$P_\gamma^{\text{Ber}} = \text{sdet}(j \circ P_\gamma \circ j^{-1})^{-1}$$

*is the parallel transport with respect to  $\nabla^{\text{Ber}}$  along the same path.*

Parallel transport with respect to the induced connection on the cotangent bundle  $\mathcal{T}^*M$  comes with an inverse. Given the construction of the Berezinian, the formula stated is thus very natural. In fact, the connection in Lem. 5.5 was constructed such as to have this parallel transport associated. The sign in that definition can be understood as the linearisation of the aforementioned inverse.

*Proof.* We must show that  $P_\gamma^{\text{Ber}}$  as stated satisfies the differential equation for parallel transport in coordinates  $(\xi^k)$ , (18), which reads as follows.

$$\partial_t P_\gamma^{\text{Ber}} = -\partial_t(\gamma^*(\xi^l)) \cdot \left( -\text{str}(j(R(\Gamma^\xi)_l)) \right) \cdot P_\gamma^{\text{Ber}}$$

By Jacobi's formula (9), the left hand side of this equation can be written as follows.

$$\partial_t P_\gamma^{\text{Ber}} = -\text{str}(j \circ (\partial_t P_\gamma) P_\gamma^{-1} \circ j^{-1}) \cdot P_\gamma^{\text{Ber}}$$

Replacing, in that expression,  $\partial_t P_\gamma$  with the right hand side of the parallel transport equation (18) with respect to  $P_\gamma$ , the analogous equation for  $P_\gamma^{\text{Ber}}$  follows.  $\square$

*Proof of Prp. 5.4.* By hypothesis, the parallel transport  $P_\gamma$  with respect to the Levi-Civita connection along some loop  $\gamma : x \rightarrow x$  is contained in some special unitary supergroup. In particular,  $\text{sdet}(j \circ P_\gamma \circ j^{-1}) = 1$  holds. By Lem. 5.6, the holonomy group functor with respect to the induced connection  $\nabla^{\text{Ber}}$  on  $\text{Ber}M$  is trivial. Continuing analogous to the proof of Prp. 4.3, one finds a global parallel nonzero homogeneous and holomorphic section  $\omega \in \text{Ber}M$ , unique up to a complex number.  $\square$

We are now in the position to give a more precise statement and proof of our second main theorem, Thm. B.

**Theorem 5.7.** *Let  $M$  be a Calabi-Yau supermanifold and  $\omega \in \text{Ber}M$  be the trivialising section of Prp. 5.4, parallel with respect to  $\nabla^{\text{Ber}}$ . Let  $\nabla^\Delta$  denote the connection from Lem. 4.1 with respect to the operator  $\Delta = \Delta^\omega$  defined in Def. 3.1. Then  $\nabla^{\text{Ber}} = \nabla^\Delta$  agree. In particular, it holds*

$$\Delta(\partial_{\xi^k}) = \text{str} \left( j \circ \nabla_{\frac{\partial}{\partial \xi^k}} \circ j^{-1} \right)$$

*Proof.* By assumption,  $\omega$  is parallel with respect to  $\nabla^{\text{Ber}}$ . By the second part of the proof of Thm. A, it is also parallel with respect to  $\nabla^\Delta$ . Writing  $\omega = h \cdot [d\xi]$  (in coordinates  $(\xi^k)$ ), we thus find

$$(\partial_{\xi^k}(h) - h\Delta(\partial_{\xi^k})) [d\xi] = \nabla_{\partial_{\xi^k}}^\Delta \omega = 0 = \nabla_{\partial_{\xi^k}}^{\text{Ber}} \omega = \left( \partial_{\xi^k}(h) - h \text{str} \left( j \circ \nabla_{\partial_{\xi^k}} \circ j^{-1} \right) \right) [d\xi]$$

from which the formula claimed is obvious. Moreover, the connections agree.  $\square$

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## A Elements of Complex Supergeometry

This appendix is meant to serve as a short introduction to selected elements of complex supergeometry needed in the main text. Although the theory presented here is certainly not new, we consider it worth to have it collected in a uniform and consistent fashion. The definitions, conventions and results contained for easy reference allow for a more

concise form of the main text with less disruptions, as a reward for the few extra pages here.

The first subsection is rather general and applies simultaneously to the smooth and complex categories. This is different from the main text, and the remainder of this appendix. Starting with the second subsection, topics include definition and properties of the Schouten-Nijenhuis bracket, parallel transport and superholonomy and, finally, an exposition on the canonical bundle and integral forms.

## A.1 Supermanifolds and Vector Bundles

Let  $M = (M_0, \mathcal{O}_M)$  be a supermanifold in the sense of Berezin-Kostant-Leites ([Lei80], [Var04], [CCF11]) with underlying classical manifold  $M_0$ . We allow  $M$  to be either real or complex. Ignorant of some abuse of notation, a (super) vector bundle  $\mathcal{E}$  on  $M$  can be defined as a sheaf of locally free  $\mathcal{O}_M$ -supermodules. The definition implies that transition functions are smooth (holomorphic) if  $M$  is a real (complex) supermanifold. Examples include the (super) tangent bundle  $\mathcal{T}M$  which, by definition, is the sheaf of  $\mathcal{O}_M$ -superderivations.

Let  $\mathcal{E} \rightarrow N$  be a super vector bundle over  $N$  and  $\varphi : M \rightarrow N$  be a morphism of supermanifolds. The pullback of  $\mathcal{E}$  under  $\varphi$  is defined as

$$(4) \quad \varphi^* \mathcal{E}(U) := \mathcal{O}_M(U) \otimes_{\varphi} (\varphi_0^* \mathcal{E})(U), \quad U \subseteq M_0 \text{ open}$$

Here,  $\varphi_0^* \mathcal{E}$  is the pullback of the sheaf  $\mathcal{E}$  under the continuous map  $\varphi_0$  which, in terms of its sheaf space, is the bundle of stalks  $\mathcal{E}_{\varphi_0(x)}$  attached to  $x \in M_0$ . (4) indeed yields a vector bundle  $\varphi^* \mathcal{E}$  on  $M$  of rank  $\text{rk} \mathcal{E}$ . For details, consult [Han12] and [Ten75]. In the same context, there is a canonical notion of pullback  $\varphi_0^* X \in \varphi_0^* \mathcal{E}$  for a section  $X \in \mathcal{E}$ , and the extension

$$\varphi^* X := 1 \otimes_{\varphi} \varphi_0^* X \in \varphi^* \mathcal{E}$$

The definition is such that

$$\varphi^*(X \cdot f) = \varphi^* X \cdot \varphi^\sharp(f) \quad \text{for } X \in \mathcal{E}, \quad f \in \mathcal{O}_N$$

A local frame  $(T^k)$  of  $\mathcal{E}$  gives rise to a local frame  $(\varphi_0^* T^k)$  of  $\varphi_0^* \mathcal{E}$  and a local frame  $(\varphi^* T^k)$  of  $\varphi^* \mathcal{E}$  such that, locally, every section  $X \in \varphi^* \mathcal{E}$  can be written  $X = \varphi^* T^k \cdot X^k$  with  $X^k \in \mathcal{O}_M(U)$ .

In the case of the tangent bundle  $\mathcal{T}N$ , there is a canonical identification of the pullback  $\varphi^* \mathcal{T}N$  with the sheaf of derivations along  $\varphi$ , through the prescription

$$\varphi^* X \mapsto \varphi^\sharp \circ X$$

with the right hand side acting on functions  $f \in \mathcal{O}_N$ . Similarly, the pullback  $\varphi^* \mathcal{T}^* N$  of the cotangent bundle is identified with the dual  $(\varphi^* \mathcal{T}N)^*$  through

$$\varphi^* \xi \mapsto \left( \varphi^* X \mapsto (\varphi^* \xi)(\varphi^* X) := \varphi^\sharp(\xi(X)) \right) \quad \text{for } \xi \in \mathcal{T}^* N$$

In the following, we shall use the aforementioned identifications without an explicit mention.

The differential of  $\varphi$  is defined by

$$d\varphi : \mathcal{T}M \rightarrow \varphi^* \mathcal{T}N, \quad d\varphi[X] := X \circ \varphi^\sharp$$

In the case that  $\varphi$  is an isomorphism, there is a further identification of the pullback of a vector field  $X \in \mathcal{T}N$  or covector field  $\xi \in \mathcal{T}^*N$ , respectively, as follows.

$$(5) \quad \varphi^* X \mapsto \varphi^* X := \varphi^\sharp \circ X \circ (\varphi^{-1})^\sharp \in \mathcal{T}M, \quad \varphi^* \xi \mapsto \varphi^* \xi := (\varphi^* \xi) \circ d\varphi \in \mathcal{T}^*M$$

We state the following natural properties.

$$(\varphi^* \xi)(\varphi^* X) = \varphi^\sharp(\xi(X)), \quad \varphi^* df = d(\varphi^\sharp(f)) \quad \text{for} \quad f \in \mathcal{O}_N$$

An isomorphism  $\varphi : M \supseteq U \mapsto \mathbb{R}^{n|m}$  is determined by coordinates, that is superfunctions,  $(\xi^k) = (x^1, \dots, x^n, \theta^1, \dots, \theta^m)$  on  $U$ , through the pullback of the corresponding functions on  $\mathbb{R}^{n|m}$ . Given coordinates, (co)vector fields on  $U$  and (co)vector fields on  $\mathbb{R}^{n|m}$  are identified through (5). A vector field  $X \in \mathcal{T}M$  has a local expression  $X = \frac{\partial}{\partial \xi^l} \cdot X^l$  where  $\frac{\partial}{\partial \xi^k}$  corresponds to the canonical derivation on  $\mathbb{R}^{n|m}$  and  $X^l$  is a uniquely determined superfunction on  $U$ . With  $(\zeta^i)$  denoting coordinates on  $N$ , the differential of a map  $\varphi : M \rightarrow N$  has the following local expression.

$$d\varphi^i_k := (-1)^{(|\xi^k| + |\zeta^i|) \cdot |\zeta^i|} \frac{\partial \varphi^\sharp(\zeta^i)}{\partial \xi^k} \quad \text{s.th.} \quad d\varphi[X] = \sum_{i,k} \left( \varphi^\sharp \circ \frac{\partial}{\partial \zeta^i} \right) \cdot d\varphi^i_k \cdot X^k$$

Interchanging derivatives gives rise to the following formula.

$$(6) \quad \partial_j(d\varphi^m_n) = (-1)^{|n||j| + |n||m| + |j||m|} \partial_n(d\varphi^m_j)$$

We state the chain rule next. Let  $\varphi : M \rightarrow N$  and  $\psi : N \rightarrow P$  be morphisms. Then

$$(7) \quad d(\psi \circ \varphi)[X] = \left( \varphi^\sharp \circ \psi^\sharp \circ \frac{\partial}{\partial \pi^l} \right) \cdot \varphi^\sharp(d\psi^l_i) \cdot d\varphi^i_k \cdot X^k$$

with  $(\pi^l)$  coordinates on  $P$  and indices  $k, i$  referring to (unlabelled) coordinates on  $M$  and  $N$ , respectively.

Let  $\varphi$  be invertible. We write  $d\varphi^{-1} := (d\varphi)^{-1}$ . In terms of coordinates on  $M$  and  $N$ , this is related to  $d(\varphi^{-1})$  by

$$(8) \quad (d\varphi^{-1})^l_m = \varphi^\sharp \left( d(\varphi^{-1})^l_m \right)$$

as follows immediately from the chain rule. In this context, Jacobi's formula reads as follows, with  $X \in \mathcal{T}M$  of definite parity.

$$(9) \quad X(\text{sdet} d\varphi) = (-1)^{m+|X|(m+n)} \text{sdet} d\varphi \cdot (d\varphi^{-1})^m_n \cdot X(d\varphi^n_m)$$

The even case was proved as Lem. 2.4 of [Gro14a], and the odd case follows e.g. by means of introducing an additional odd generator. (9) has the following corollary.

**Lemma A.1.** *Let  $\varphi : M \rightarrow N$  be invertible. Then*

$$\sum_j \partial_j \left( \text{sdet} d\varphi \cdot (d\varphi^{-1})^j_k \right) = 0$$



*Proof.* We use (9) to calculate term (i) in

$$\partial_j(\text{sdet}d\varphi(d\varphi^{-1})^j_k) = \partial_j(\text{sdet}d\varphi) \cdot (d\varphi^{-1})^j_k + \text{sdet}d\varphi \cdot \partial_j(d\varphi^{-1})^j_k =: (i) + (ii)$$

as follows, using (6).

$$\begin{aligned} (i) &= (-1)^{n+jn+jm} \text{sdet}d\varphi \cdot (d\varphi^{-1})^n_m \partial_j(d\varphi^m_n)(d\varphi^{-1})^j_k \\ &= (-1)^{n+nm} \text{sdet}d\varphi \cdot (d\varphi^{-1})^n_m \partial_n(d\varphi^m_j)(d\varphi^{-1})^j_k \\ &= -\text{sdet}d\varphi \cdot \partial_n(d\varphi^{-1})^n_m d\varphi^m_j (d\varphi^{-1})^j_k \\ &= -\text{sdet}d\varphi \cdot \partial_n(d\varphi^{-1})^n_k \\ &= -(ii) \end{aligned}$$

□

Let  $\varphi : \mathbb{R}^{n|m} \rightarrow \mathbb{R}^{n|m}$  be a change of coordinates  $\zeta^j = \varphi^\sharp(\xi^j)$ , symbolically denoted  $\varphi : \zeta \rightarrow \xi$ . By construction, the transformation of coordinate expressions of (co)vectors under  $\varphi$  is governed by (5). By means of the local expression of the differential, this can be rewritten

$$(10) \quad \varphi^* \partial_{\xi^k} = \partial_{\zeta^m} \cdot (d\varphi^{-1})^m_k, \quad \varphi^* d\xi^j = (-1)^{j+jm} d\zeta^m \cdot d\varphi^j_m = d\zeta^m \cdot (d\varphi^{ST})^m_j$$

Here,  $d : \mathcal{O}_M \rightarrow \mathcal{T}^*M$  is the differential on superfunctions, which we define with the convention

$$df[X] := (-1)^{|f||X|} X(f), \quad \text{for } X \in \mathcal{T}M$$

As usual, the superscript "ST" in (10) means supertranspose, and the indices refer to the local basis elements  $(d\xi^m)$  and  $(d\zeta^k)$  of  $\mathcal{T}^*M$  induced by the coordinate functions. Nota bene, such a basis is related to the standard dual basis of the coordinate vector fields by  $(\partial_{\xi^m})^* = (-1)^m d\xi^m$ .

## A.2 Complex Supermanifolds and Differential Forms

In the following, we consider a complex supermanifold  $M = (M_{\overline{0}}, \mathcal{O}_M)$  of dimension  $n|m$ , which we always assume connected. By a standard construction (see [HW87]),  $M$  carries a canonical structure of a real supermanifold of dimension  $2n|2m$ , which is compatible with the real manifold of dimension  $2n$  associated with  $M_{\overline{0}}$ . In terms of (complex) local coordinates  $(\xi^k)$ , the picture is as follows. With a natural notion of conjugating superfunctions, the relations

$$(11) \quad \xi_R^k = \frac{1}{2}(\xi^k + \bar{\xi}^k), \quad \xi_I^j = \frac{1}{2i}(\xi^j - \bar{\xi}^j)$$

define real coordinates  $(\xi_R^k, \xi_I^j)$ . Likewise, a morphism  $\varphi : M \rightarrow N$  between complex supermanifolds canonically induces a morphism between the corresponding real supermanifolds. We shall use the resulting functor frequently in an implicit manner, writing  $M$  and  $\varphi$  for both the complex or real supermanifold or morphism, respectively. An analogous comment applies to the sheaves of holomorphic and real superfunctions. By a slight abuse of notation, we shall write  $\mathcal{O}_M$  in either case.

We define a holomorphic (super) vector bundle on  $M$  to be a vector bundle in the general sense of the previous subsection, on  $M$  considered as a complex supermanifold. Likewise, a smooth (super) vector bundle on  $M$  is a vector bundle on  $M$  considered as a real supermanifold. There is a forgetful functor from holomorphic vector bundles to smooth vector bundles.

The tangent bundle of a supermanifold, complex or real, was defined in the previous subsection as the sheaf of superderivations with respect to the sheaf of superfunctions. In the present context, there are two variants thereof, according to the complex or real pictures of  $M$ . We adopt the following convention. The real and complex tangent bundles shall be denoted by  $\mathcal{T}M$  and  $\mathcal{T}^{1,0}M$ , respectively. They are related to each other, analogous to the case of classical complex manifolds, as follows. The complex structure of  $M$  induces a complex structure  $J$  on the smooth bundle  $\mathcal{T}M$ . The eigensheaf with eigenvalue  $+1$  of the complexification  $\mathcal{T}M \otimes \mathbb{C}$  can be identified with the complex tangent bundle  $\mathcal{T}^{1,0}M$ . As usual, we denote the  $-1$ -eigensheaf by  $\mathcal{T}^{0,1}M$ . There is a canonical complex-linear isomorphism

$$(12) \quad j : (\mathcal{T}M, J) \rightarrow (\mathcal{T}^{1,0}M, i), \quad X \mapsto \frac{1}{2}(X - iJX)$$

In terms of complex coordinates  $(\xi^k)$  with induced real coordinates (11), the respective canonical derivations are related by

$$(13) \quad j\left(\partial_{\xi_R^k}\right) = \partial_{\xi^k}, \quad j\left(\partial_{\xi_I^k}\right) = i\partial_{\xi^k}$$

In the rest of this subsection, we state some properties of bundles of differential forms. Up to signs, this part largely parallels the ungraded case and is, therefore, kept short. The prescription

$$\Omega^{p,q}M := \bigwedge^p (\mathcal{T}^{1,0}M)^* \otimes \bigwedge^q (\mathcal{T}^{0,1}M)^*$$

defines smooth vector bundles over  $M$  which are referred to as bundles of differential forms. On  $\Omega^{p,q}M$ , there is a natural notion of exterior derivative, and its canonical projection to  $\Omega^{p,q+1}M$  is denoted  $\bar{\partial}$ . We are mostly interested in the case  $p = 0$ . A section can then locally be written as a sum of elements of the form  $d\bar{\xi}^I \cdot f$ , where  $(\xi^k)$  are complex coordinates of  $M$  with conjugation as in (11), and  $I$  denotes some multiindex. The  $\bar{\partial}$ -operator then reads

$$\bar{\partial}\left(d\bar{\xi}^I \cdot f\right) = \sum_k \left( (-1)^{|\xi^k||\xi^I|} d\bar{\xi}^k \wedge d\bar{\xi}^I \cdot \frac{\partial f}{\partial \xi^k} \right)$$

We also consider differential forms with values in some holomorphic vector bundle  $\mathcal{E}$ , that is bundles of the form

$$\Omega^{0,q}(\mathcal{E}) := \Omega^{0,q}M \otimes \mathcal{E}$$

Define the  $\bar{\partial}$ -operator on such bundles via

$$(14) \quad \bar{\partial}(\alpha \otimes e) := \bar{\partial}(\alpha) \otimes e$$

This is well-defined, since  $e$  transforms holomorphically (annihilated by  $\bar{\partial}$ ).

Of particular interest will be multivector fields  $\mathcal{E} = \bigwedge^p \mathcal{T}^{1,0} M$ . Throughout, we adopt the degree conventions explained in Sec. 2. In particular, the cohomological degree of a section  $\alpha \in \Omega^{0,q}(\bigwedge^p \mathcal{T}^{1,0} M)$  is  $\deg(\alpha) = p + q$ . The Schouten-Nijenhuis bracket, to be defined next, is a natural supersisation of the classical bracket bearing that name (see e.g. Chp. 6 of [Huy05] for a formula) It extends the vector field bracket.

**Definition A.2** (Schouten-Nijenhuis bracket). *Let  $[\cdot, \cdot]$  be the map*

$$[\cdot, \cdot] : \Omega^{0,q} \left( \bigwedge^p \mathcal{T}^{1,0} M \right) \times \Omega^{0,q'} \left( \bigwedge^{p'} \mathcal{T}^{1,0} M \right) \longrightarrow \Omega^{0,q+q'} \left( \bigwedge^{p+p'-1} \mathcal{T}^{1,0} M \right)$$

*defined as follows. For  $f \in \mathcal{O}_M$  and  $v_i, w_j \in \mathcal{T}^{1,0} X$ , we set*

$$\begin{aligned} [f, v_1 \wedge \dots \wedge v_p] &:= -(-1)^{(i-1)+|v_i|(\sum_{l=1}^{i-1} |v_l| + |f|)} v_i(f) \cdot v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \\ [v_1 \wedge \dots \wedge v_p, f] &:= -(-1)^{(p+1)+|f|(|v_1|+\dots+|v_p|)} [f, v_1 \wedge \dots \wedge v_p] \end{aligned}$$

*and*

$$\begin{aligned} [w_1 \wedge \dots \wedge w_{p'}, v_1 \wedge \dots \wedge v_p] \\ := \sum_{j,i} (-1)^{M(i,j)} [w_j, v_i] \wedge (w_1 \wedge \dots \wedge \widehat{w_j} \wedge \dots) \wedge (v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots) \end{aligned}$$

*with the sign*

$$M(i,j) := i + j + |w_j| \sum_{l=1}^{j-1} |w_l| + |v_i| \left( \sum_{l=1}^{i-1} |v_l| + \sum_{l=1}^{p'} |w_l| + |w_j| \right)$$

*With  $(\xi^k)$  local coordinates and  $I, J, K, L$  denoting multiindices, we further prescribe*

$$\begin{aligned} [f d\bar{\xi}^I \otimes \partial_{\xi^J}, g d\bar{\xi}^K \otimes \partial_{\xi^L}] \\ := (-1)^{q'(p+1)+|f||\xi^I|+|\xi^K|(|g|+|\xi^J|+|f|)} d\bar{\xi}^I \wedge d\bar{\xi}^K \otimes [f \partial_{\xi^J}, g \partial_{\xi^L}] \end{aligned}$$

*where  $q = |I|$ ,  $p = |J|$ ,  $q' = |K|$ ,  $p' = |L|$ .*

Although written in terms of coordinates  $(\xi^k)$  on  $M$ , the resulting bracket is well-defined since the transformation of  $d\bar{\xi}^I$  gets annihilated by  $v \in \mathcal{T}^{1,0} M$ .

**Lemma A.3.** *The Schouten-Nijenhuis bracket satisfies the following symmetry.*

$$[\alpha, \beta] = -(-1)^{(\deg(\alpha)+1)(\deg(\beta)+1)+|\alpha||\beta|} [\beta, \alpha]$$

*Proof.* This follows from the definition by a straightforward calculation. □

**Definition A.4.** *Exterior product:*

$$\Omega^{0,p} \left( \bigwedge^r \mathcal{T}^{1,0} M \right) \times \Omega^{0,q} \left( \bigwedge^s \mathcal{T}^{1,0} M \right) \rightarrow \Omega^{0,p+q} \left( \bigwedge^{r+s} \mathcal{T}^{1,0} M \right)$$

*defined as follows, for  $\bar{\alpha} \in \Omega^{0,p} M$ ,  $\beta \in \Gamma(\bigwedge^r \mathcal{T}^{1,0} M)$ ,  $\bar{\gamma} \in \Omega^{0,q} M$  and  $\delta \in \bigwedge^s \mathcal{T}^{1,0} M$ .*

$$(\bar{\alpha} \otimes \beta) \wedge (\bar{\gamma} \otimes \delta) := (-1)^{r q + |\beta| |\bar{\gamma}|} (\bar{\alpha} \wedge \bar{\gamma}) \otimes (\beta \wedge \delta)$$

**Lemma A.5.** *Let  $\alpha \in \Omega^{0,q}(\bigwedge^p \mathcal{T}^{1,0}M)$ . The map*

$$[\alpha, \cdot] : \Omega^{0,q'} \left( \bigwedge^{p'} \mathcal{T}^{1,0}M \right) \longrightarrow \Omega^{0,q+q'} \left( \bigwedge^{p+p'-1} \mathcal{T}^{1,0}M \right)$$

*is a derivation. More precisely,*

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(\deg(\alpha)+1)\deg(\beta)+|\alpha||\beta|} \beta \wedge [\alpha, \gamma]$$

*Proof.* This follows from a lengthy, but otherwise straightforward, calculation in local expressions, using the definitions. One proves the case  $\alpha, \beta, \gamma \in \Omega^{0,0}(\bigwedge \mathcal{T}^{1,0}M)$  first and deduces from this the general case.  $\square$

### A.3 Connections, Parallel Transport and Holonomy

Let  $M$  continue to denote a complex supermanifold with associated real supermanifold referred to by  $M$ , too. A connection on a smooth vector bundle  $\mathcal{E}$  on  $M$  is an even real-linear sheaf morphism

$$\nabla : \mathcal{E} \rightarrow \mathcal{T}M^* \otimes_{\mathcal{O}_M} \mathcal{E}, \quad \nabla(fe) = df \otimes_{\mathcal{O}_M} e + f \cdot \nabla e \quad \text{for } f \in \mathcal{O}_M$$

with  $\mathcal{O}_M$  denoting the real superfunctions. In the case of a holomorphic vector bundle, we refer to a connection as a connection in the above sense on the associated smooth vector bundle. In particular, in our point of view, we do not demand complex linearity but, if present, consider it as an extra structure.

A connection  $\nabla$  on the tangent bundle  $\mathcal{E} = \mathcal{T}M$  induces a connection on  $\mathcal{T}^{1,0}M$ , in the following referred to by the same symbol, through the prescription

$$(15) \quad X \mapsto j \circ \nabla \circ j^{-1}(X)$$

involving the canonical isomorphism (12). This connection is complex linear precisely if  $\nabla J = 0$  (Kähler case). In this case, it coincides with the complex-linear extension  $\nabla^{\mathbb{C}}$  restricted to  $\mathcal{T}^{1,0}M \subseteq \mathcal{T}^{\mathbb{C}}M$ .

We shall need the transformation behaviour of a connection  $\nabla$  on  $\mathcal{T}M$  under a transformation of coordinates  $\varphi : \zeta \rightarrow \xi$ . In terms of coordinates  $(\xi^i)$ , the connection is determined by its left or right Christoffel symbols

$$(16) \quad \nabla_{\partial_{\xi^k}} \partial_{\xi^l} = \Gamma_{kl}^q \cdot \partial_{\xi^q} = \partial_{\xi^q} \cdot {}^R \Gamma_{kl}^q$$

The pullback under a (local) isomorphism  $\varphi$ , as demanded by (5), is defined as follows

$$(17) \quad (\varphi^* \nabla)_{\varphi^* X}(\varphi^* Y) := \varphi^*(\nabla_X Y)$$

By the discussion leading to (10), equation (17) describes the transformation of the local expression for  $\nabla$  in case  $\varphi : \zeta \rightarrow \xi$  is a change of coordinates. The Christoffel symbols with respect to  $\zeta$  are then given by

$$\hat{\Gamma}_{bc}^a \cdot \partial_{\zeta^a} = (\varphi^* \nabla)_{\partial_{\zeta^a}} \partial_{\zeta^b}$$

They are related to  $\Gamma$  by the following result.

**Lemma A.6.** *Let  $\nabla$  be a connection on  $\mathcal{TM}$  and  $\varphi : \zeta \rightarrow \xi$  be a coordinate transformation. Then, with the notation introduced above,*

$$\varphi^\#(\Gamma_{kl}^p) = (-1)^{m(m+k)+ql+p(p+q)} \\ (d\varphi^{-1})^m_k \cdot \left( (-1)^{qn} \hat{\Gamma}_{mn}^q (d\varphi^{-1})^n_l + (-1)^q \partial_{\xi^m} (d\varphi^{-1})^q_l \right) \cdot (d\varphi)^p_q$$

*Proof.* This follows from a straightforward calculation involving (10) and Leibniz' rule.  $\square$

Returning to the general case of a connection  $\nabla$  on a vector bundle  $\mathcal{E}$ , we mention the notion of parallel transport  $P_\gamma : x^*\mathcal{E} \rightarrow y^*\mathcal{E}$  along an  $S$ -path  $\gamma : S \times [0, 1] \rightarrow M$  connecting  $S$ -points  $x, y : S \rightarrow M$ . Here,  $S$  is a parameter space which we restrict to the case of superpoints  $S = \mathbb{R}^{0|L}$ . This is detailed in [Gro14b]. In terms of local coordinates  $(\xi^k)$  and a trivialisation  $(T^m)$  of  $\mathcal{E}$ , the parallelness condition  $(\gamma^*\nabla)_{\partial_t} P_\gamma[X_x] = 0$  for  $X_x \in x^*\mathcal{E}$  reads as follows.

$$(18) \quad \partial_t(P_\gamma)^m_p = -(-1)^{m(k+1)} \partial_t(\gamma^*(\xi^l)) \cdot \gamma^*(\Gamma_{lk}^m) \cdot (P_\gamma)^k_p$$

In this equation, the Christoffel symbols are defined analogous to (16).

**Lemma A.7.** *Let  $P_\gamma$  denote parallel transport of a connection on  $\mathcal{TM}$  along some  $S$ -path  $\gamma$ . Then parallel transport with respect to the induced connection (15) on  $\mathcal{T}^{1,0}$  is given by  $j \circ P_\gamma \circ j^{-1}$ .*

A meaningful notion of holonomy in supergeometry was first introduced in [Gal09] via a Harish-Chandra superpair construction, while a categorical approach was developed in [Gro14b]. As analysed in [Gro16], both theories are equivalent, although in a nontrivial fashion through the Twofold Theorem and the Comparison Theorem established in that reference. In this article, we utilise the categorical approach that we shall sketch in the following.

For a fixed  $S$ -point  $x$ , the holonomy group  $\text{Hol}_x$  is defined as the set of parallel transport operators  $P_\gamma$  with  $\gamma$  a piecewise smooth  $S$ -loop starting and ending in  $x$ . It can be shown to carry the structure of a Lie group. By a theorem of Ambrose-Singer type, its Lie algebra  $\text{hol}_x$  is generated by endomorphisms of the form

$$(19) \quad \{P_\gamma^{-1} \circ R_y(u, v) \circ P_\gamma \mid y : S \rightarrow M, \gamma : x \rightarrow y \text{ pw.smooth}, u, v \in (y^*\mathcal{TM})_{\bar{0}}\}$$

where  $R$  denotes the curvature tensor with respect to  $\nabla$ . In particular,  $\text{Hol}_x$  is trivial for a flat connection (with  $R = 0$ ) on a simply-connected supermanifold.

As it stands,  $\text{Hol}_x$  alone does not contain enough information for a good holonomy theory. To that end, let  $T = \mathbb{R}^{0|L'}$  be another superpoint and consider  $x$  as an  $S \times T$ -point, denoted  $x_T : S \times T \rightarrow M$ . The prescription  $T \mapsto \text{Hol}_x(T) := \text{Hol}_{x_T}$  extends to a Lie group valued functor, referred to as the holonomy group functor  $\text{Hol}_x$ . The reader should note that, in general, this functor is not representable. In particular, it cannot be identified with the  $\Lambda$ -point functor of Schwarz [Shv84] and Voronov [Vor84], by means of which a supermanifold is characterised.

For the purposes of the present article it suffices to consider the holonomy group functor with respect to a topological point  $x : \mathbb{R}^{0|0} \rightarrow M$ . The Holonomy Principle can then be cast in the following form. Recall that all supermanifolds occurring are assumed connected.

**Theorem A.8** (Holonomy Principle). *The pullback  $X_x := x^*X \in x^*\mathcal{E}$  of a parallel global section  $X \in \mathcal{E}$  with  $\nabla X \equiv 0$  is holonomy invariant  $\text{Hol}_x(T) \cdot X_x = X_x$  for every  $T = \mathbb{R}^{0|L'}$ . Conversely, invariance in this sense of a section  $X_x \in x^*\mathcal{E}$  implies the unique existence of a parallel global section  $X \in \mathcal{E}$  such that  $x^*X = X_x$ .*

#### A.4 The Canonical Bundle and Integral Forms

The Berezinian of a free  $A$ -supermodule  $M$  of rank  $p|q$  for  $A$  a supercommutative superalgebra is the free  $A$ -supermodule of rank  $1|0$  (for  $q$  even) or  $0|1$  (for  $q$  odd) defined through a distinguished class of bases  $[s] = [s_1 \dots s_{p+q}]$  for  $(s_i)$  an  $M$ -basis and the relation  $[s'] = [s] \cdot \text{Ber}\varphi$  if  $(s'_i) = (s_i) \cdot \varphi$ , see e.g. Chp. 3 of [Man88] for details. This construction carries over to vector bundles through local trivialisations. The essential notion for the present paper is the following.

**Definition A.9.** *Let  $M$  be a complex supermanifold. Its canonical bundle is the Berezinian of the complex cotangent sheaf  $\text{Ber}M := \text{Ber}(\mathcal{T}^{1,0}M)^*$ .*

As mentioned above, any choice of coordinates  $(\xi^k)$  on  $M$  induces a local covector basis  $(d\xi^k)$  of  $(\mathcal{T}^{1,0}M)^*$ . We shall use the following notation for the induced Berezinian section.

$$[d\xi] := [d\xi^1 \dots d\xi^{p+q}]$$

By (10), this transforms under a coordinate transformation  $\varphi : \zeta \rightarrow \xi$  as follows.

$$\varphi^*[d\xi] := [\varphi^*d\xi] = [d\zeta \cdot d\varphi^{ST}] = \text{sdet}d\varphi^{ST} \cdot [d\zeta] = \text{sdet}d\varphi \cdot [d\zeta]$$

**Definition A.10.** *The sheaves of integral forms are defined by*

$$I^{n-p} := \bigwedge^p \mathcal{T}^{1,0}M \otimes \text{Ber}M$$

for  $p \geq 0$ .

There is a real counterpart of the previous definition, which plays a role for the theory of integration similar to that of the de Rham-complex in classical geometry, see e.g. Chp. 3 of [DM99]. For the present complex case, there is no immediate reference to integration. Nevertheless, we refer to the  $I^{n-p}$  as 'integral' forms for the otherwise analogous structure. There is a natural operator as follows.

**Definition A.11.** *We define  $\partial : I^{n-p} \rightarrow I^{n-p+1}$  by*

$$\begin{aligned} \partial \left( f \cdot \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \otimes [d\xi] \right) \\ := \sum_{i=1}^{n+m} (-1)^{M_i} \frac{\partial f}{\partial \xi^i} \cdot \left( \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^{i-1}} \wedge \widehat{\frac{\partial}{\partial \xi^i}} \wedge \frac{\partial}{\partial \xi^{i+1}} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \right) \otimes [d\xi] \end{aligned}$$

with respect to (complex) local coordinates  $\xi^i$  on  $M$ . As is usual, the hat symbol means omission. Moreover, the sign  $M_i$  reads

$$M_i = (i-1) + |\xi^i| \cdot \left( \sum_{j=1}^{i-1} |\xi^j| + |f| \right)$$

as arising from moving  $\frac{\partial}{\partial \xi^i}$  to the front.

**Lemma A.12.**  $\partial$  is well-defined (independent of coordinates), and  $\partial^2 = 0$ .

The operator  $\partial$  should be thought of as a suitable generalisation of the classical operator  $\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}$  (with  $q = 0$ ) on a complex manifold. Indeed, in this case there is a canonical isomorphism  $I^{n-p} \cong \Omega^{p,0}$ , and the local expression for  $\delta$  is as stated. In the case at hand of a complex supermanifold, the bundles  $I^{n-p} \neq \Omega^{q,0}$  are different, and a welldefined operator of the local form stated can only be defined on the former, in the way presented here. Lem. A.12 can be proved by a lengthy but direct calculation involving the explicit expressions under a coordinate transformation of the objects occurring, and using Lem. A.1.

Let  $\bar{\partial}$  denote the operator on  $\Omega^{0,q}(I^{n-l})$  as defined by (14). Moreover, we define

$$(20) \quad \partial : \Omega^{0,q}(I^{n-p}) \rightarrow \Omega^{0,q}(I^{n-p+1}), \quad \partial(d\bar{\xi}^I \otimes e) := (-1)^q d\bar{\xi}^I \otimes \partial e$$

where  $I$  is a multiindex with  $|I| = q$  and  $e \in I^{n-p}$ . The operator  $\partial$  on the right hand side is the map from Def. A.11. Although written in terms of coordinates  $(\xi^k)$  on  $M$ , the resulting operator is welldefined since the transformation of  $d\bar{\xi}^I$  gets annihilated by  $\partial$ . It follows immediately that

$$(21) \quad \partial^2 = 0$$

**Lemma A.13.**  $\partial$  anticommutes with  $\bar{\partial}$ :

$$\partial\bar{\partial} = -\bar{\partial}\partial$$

We remark that the sign  $(-1)^q$  in (20) is important for this lemma to hold.

*Proof.* This follows from a straightforward calculation involving the expressions of either operator in local coordinates.  $\square$

We close this appendix with a comparison to Manin's theory of integral forms as presented in Sec. 4.5 of [Man88]. It turns out that the operator  $\delta$  defined in paragraph 4 of that reference coincides with our operator  $\partial$ , upon considering suitable identifications as to be detailed in the following. In particular, Lem. A.12 then follows from analogous properties of  $\delta$ .

To begin with let, in general,  $A$  be a supercommutative superalgebra and  $V$  be an  $A$ -supermodule. Let  $\Pi V$  and  $V\Pi$  denote the supermodules with reversed parity and, respectively, same right and left  $A$ -multiplication as  $V$ . Let  $TV$  denote the tensor algebra of  $V$ . We define a map  $\Gamma : TV \rightarrow T(V\Pi)$  through the prescription

$$\Gamma(a) := a, \quad \Gamma(v_1 \otimes \dots \otimes v_p) := (-1)^{\sum_{i=1}^p (p-i+1)|v_i|} (v_1\Pi) \otimes \dots \otimes (v_p\Pi)$$

for  $a \in A$  and  $v_i \in V$ . This is well-defined,  $A$ -linear and bijective. Restricting to the exterior algebra yields a well-defined map, still denoted  $\Gamma$ , as follows.

$$\Gamma : \bigwedge(V) \rightarrow S(V\Pi), \quad |\Gamma|_{\bigwedge^p V} = \sum_{i=1}^p |v_i| + \sum_{i=1}^p (|v_i| + 1) = p$$

This map is an isomorphism (in the weaker sense of non-parity preserving) of  $A$ -supermodules, but not of superalgebras. We may form the tensor product of either side with another  $A$ -module  $W$  and consider the trivial extension of  $\Gamma$ , that we shall denote by the same symbol.



In our case of interest, this yields the bundle map

$$\Gamma : \text{Ber}M \otimes \bigwedge \mathcal{T}^{1,0}M \rightarrow \text{Ber}M \otimes S(\mathcal{T}^{1,0}M\Pi)$$

defined through

$$\begin{aligned} \Gamma \left( [d\xi] \cdot f \otimes \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \right) \\ = (\text{id} \otimes \Gamma) \left( [d\xi] \cdot f \otimes \frac{\partial}{\partial \xi^1} \wedge \dots \wedge \frac{\partial}{\partial \xi^p} \right) \\ = (-1)^{p(m+|f|)+\sum_{i=1}^p(p-i+1)|\xi^i|} [d\xi] \cdot f \otimes \frac{\partial}{\partial \xi^1} \Pi \odot \dots \odot \frac{\partial}{\partial \xi^p} \Pi \end{aligned}$$

where  $m = |[d\xi]|$  is the odd dimension  $\dim_{\mathbb{C}} M = n|m$ , and the sign  $p(m+|f|)$  comes from commuting  $\Gamma$  past  $[d\xi] \cdot f$ . On the right hand side, there are the integral forms (in the complex case) in the sense of [Man88]. On the left, we have the integral forms as in Def. A.10, with the general canonical isomorphism  $V \otimes W \cong W \otimes V$  for  $A$ -supermodules  $V$  and  $W$ , defined through  $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ , left implicit.

**Lemma A.14.** *Let  $\delta : \text{Ber}M \otimes S(\mathcal{T}^{1,0}M\Pi) \odot$  denote the operator defined in Sec.4.5 of [Man88]. Then  $\delta \circ \Gamma = -\Gamma \circ \partial$ .*

*Proof.* Both operators are defined in terms of local coordinates. By a direct calculation, they coincide.  $\square$

As detailed in [Man88], the operator  $\delta$  is welldefined and satisfies  $\delta^2 = 0$ . By the preceding lemma, it is clear that  $\partial$  has analogous properties. This gives a proof of Lem. A.12.

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